MMAT3560 200217

Lecture notes Example 2.3.5 P.40-42

max $f(y_1, y_2, y_3) = \underline{y_1 + 2y_2 - y_3} + \underline{\underline{3}}$ $\qquad 5 + 2\times 4 - 0 + 3 = 16$

subj. to
$$2y_1 + y_2 + y_3 \le 14 \qquad 2\times 5 + 4 + 0 = 14 \le 14$$
$$4y_1 + 2y_2 + 3y_3 \le 34 \qquad 4\times 5 + 2\times 4 + 3\times 0 = 28 \le 34$$
$$2y_1 + 5y_2 + 5y_3 \le 30 \qquad 2\times 5 + 5\times 4 + 5\times 0 = 30 \le 30$$
$$y_1, y_2, y_3 \ge 0$$

Sol. Primal Problem

$$\boxed{A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 2 & 3 \\ 2 & 5 & 5 \end{pmatrix}, \quad \vec{b} = (14, 34, 30), \quad \vec{c} = (1, 2, -1), \quad d = 3}$$

Genuine variables $\qquad$ Slack variables

$\vec{y} = (y_1, y_2, y_3) \qquad \vec{y}_s = (y_4, y_5, y_6)$

$\vec{x} = (x_1, x_2, x_3) \qquad \vec{x}_s = (x_4, x_5, x_6)$

Step 1. Tableau

	y_1	y_2	y_3	-1
x_1	2^*	1	1	14 $= -y_4$

$\Leftrightarrow 2y_1 + y_2 + y_3 - 14 = -y_4$

x_2	4	2	3	34	$= -y_5$ $\frac{1}{2}$
x_3	2	5	5	30	$= -y_6$ 15
-1	1	2	-1	-3	$= f$
	$= x_4$	$= x_5$	$= x_6$	$= g$	

$$2y_1 + 5y_2 + 5y_3 - 30 = -y_6$$

$$y_1 + 2y_2 - y_3 + 3 = f$$

$$x_1 + 2x_2 + 5x_3 - 2 = x_5$$

$\downarrow$

	$\overset{y_4}{x_1}$	y_2	y_3	-1	
x_4 y_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	7	$= -y_1$ 14
x_2	-2	0	1	6	$= y_5$
x_3	-1	4^*	4	16	$- y_6$ 4
-1	$-\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	-10	$= f$
	$= x_1$	$= x_5$	$= x_6$	$= g$	

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} \frac{1}{a} & \frac{b}{a} \\ -\frac{c}{a} & \underline{d - \frac{bc}{a}} \end{pmatrix}$$

$$d - \frac{bc}{a} = 7 - \frac{16 \times \frac{1}{2}}{4} = 5$$

$$d - \frac{bc}{a} = -10 - \frac{16 \times \frac{3}{2}}{4} = -16$$

$$d - \frac{bc}{a} = \frac{1}{2} - \frac{-1 \times \frac{1}{2}}{4} = \frac{5}{8}$$

	$\overset{y_4}{x_1}$	$\overset{y_6}{x_3}$	y_3	-1	
x_4 y_1	$\frac{5}{8}$	$-\frac{1}{8}$	0	5	$= -y_1$
x_2	-2	0	1	6	$= -y_5$

x_5 y_2	-4	4	1	1	y_2
	$-\frac{1}{8}$	$-\frac{3}{8}$	-3	-16	
	x_1	x_3	x_6		

Optimal solution:

$\vec{y} = (y_1, y_2, y_3) = (5, 4, 0)$ $\quad$ $\vec{y_s} = (y_4, y_5, y_6) = (0, 6, 0)$

max. value of $f(\vec{y}) = 16$

Dual Problem:

$\vec{x} = (x_1, x_2, x_3) = (\frac{1}{8}, 0, \frac{3}{8})$, $\vec{x_s} = (x_4, x_5, x_6) = (0, 0, 3)$

min $\quad g(x_1, x_2, x_3) = 14x_1 + 34x_2 + 30x_3 + 3$

Subj. to
$$2x_1 + 4x_2 + 2x_3 \geq 1$$
$$x_1 + 2x_2 + 5x_3 \geq 2$$
$$x_1 + 3x_2 + 5x_3 \geq -1$$
$$x_1, x_2, x_3 \geq 0$$

$g(\frac{1}{8}, 0, \frac{3}{8}) = 16$

min. of value of $g(\vec{x}) = 16$

Transforming Minimax Problem to Primal Problem

Step1. Add a constant $\underline{\underline{k}}$ to every entry of A so that all entries are positive

Step 2. Solve the primal and dual problem

$$\begin{array}{c|c|c} & \vec{y} & \\ \hline \vec{x}^T & A & \vec{1}^T \\ \hline & \vec{1} & 0 \end{array}$$

$\vec{b} = \vec{1} = (1, 1, \cdots, 1) \in \mathbb{R}^m$

$\vec{c} = \vec{1} = (1, 1, \cdots, 1) \in \mathbb{R}^n$

$d = 0$

Change of variable: $\vec{x} = \frac{1}{v}\vec{p}$, $\vec{y} = \frac{1}{v}\vec{q}$

v = value of A

Step 3. Write down solution

maximin strategy: $\vec{p} = \frac{1}{d}\vec{x}_0$

minimax strategy: $\vec{q} = \frac{1}{d}\vec{y}_0$

value of A : $v = \frac{1}{d} - k$

Example: lecture notes Example 2.3.7 P 44-45

Example: [illegible]

$$\boxed{A = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & -2 \\ 0 & 1 & -1 \end{pmatrix}}$$

Solution. Add $k=2$ to every entry

$$A' = \begin{pmatrix} 5 & 2 & 3 \\ 1 & 4 & 0 \\ 2 & 3 & 1 \end{pmatrix} \qquad \begin{aligned} \vec{b} &= (1,1,1) \\ \vec{c} &= (1,1,1) \\ d &= 0 \end{aligned}$$

$$\begin{array}{c|ccc|c|l}
 & y_1 & y_2 & y_3 & -1 & \\
\hline
x_1 & 5^{*} & 2 & 3 & 1 & = -y_4 \quad \frac{1}{5} \\
x_2 & 1 & 4 & 0 & 1 & = -y_5 \quad 1 \\
x_3 & 2 & 3 & 1 & 1 & = -y_6 \quad \frac{1}{2} \\
\hline
-1 & 1 & 1 & 1 & 0 & \\
 & \shortparallel & \shortparallel & \shortparallel & & \\
 & x_4 & x_5 & x_6 & &
\end{array}$$

$$\downarrow$$

$$\begin{array}{c|ccc|c|l}
 & x_1 & y_2 & y_3 & & \\
\hline
y_1 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{1}{5} & \quad \frac{1}{2} \\
x_2 & -\frac{1}{5} & \frac{18}{5}^{*} & -\frac{3}{5} & \frac{4}{5} & \quad \frac{2}{9}
\end{array}$$

x_3	$-\frac{2}{5}$	$\frac{11}{5}$	$-\frac{1}{5}$	$\frac{3}{5}$	$\frac{3}{11}$
	$-\frac{1}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	

$\downarrow$

	x_1	x_2	y_3		
y_1	$\frac{2}{9}$	$-\frac{1}{9}$	$\frac{2}{3}$*	$\frac{1}{9}$	$\frac{1}{6}$
y_2	$-\frac{1}{18}$	$\frac{5}{18}$	$-\frac{1}{6}$	$\frac{2}{9}$	
x_3	$-\frac{5}{18}$	$-\frac{11}{18}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{2}{3}$
	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{3}$	

$\downarrow$

		y_4	y_5			
		x_1	x_2	y_1	-1	
x_6	y_3	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{3}{2}$	$\frac{1}{6}$	$= -y_3$
x_5	y_2	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$= -y_2$
	x_3	$-\frac{1}{3}$	$-\frac{17}{12}$	$-\frac{1}{4}$	$\frac{1}{12}$	$= y_6$
	-1	$-\frac{1}{3}$	$-\frac{1}{12}$	$-\frac{3}{4}$	$-\frac{5}{12}$	
		$\|$	$\|$	$\|$		

$$-\frac{1}{18} - \frac{\frac{2}{9} \times \left(-\frac{1}{6}\right)}{\frac{2}{3}} = 0$$

$x_1 \quad x_2 \quad x_4$

$d = \frac{5}{12}$ ✗ value of $A' = \frac{1}{d} = \frac{12}{5}$

$\vec{x}_0 = (x_1, x_2, x_3) = (\frac{1}{3}, \frac{1}{12}, 0)$ $\vec{x}_s = (x_4, x_5, x_6) = (\frac{3}{4}, 0, 0)$

$\vec{y}_0 = (y_1, y_2, y_3) = (0, \frac{1}{4}, \frac{1}{6})$ $\vec{y}_s = (y_4, y_5, y_6) = (0, 0, \frac{1}{12})$

maximin strategy: $\vec{p} = \frac{1}{d}\vec{x}_0 = \frac{12}{5}(\frac{1}{3}, \frac{1}{12}, 0) = (\frac{4}{5}, \frac{1}{5}, 0)$

minimax strategy: $\vec{q} = \frac{1}{d}\vec{y}_0 = \frac{12}{5}(0, \frac{1}{4}, \frac{1}{6}) = (0, \frac{3}{5}, \frac{2}{5})$

Value of A: $v = \frac{1}{d} - k = \frac{12}{5} - 2 = \frac{2}{5}$

Check: $\vec{p}A = (\frac{4}{5}, \frac{1}{5}, 0)\begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & -2 \\ 0 & 1 & -1 \end{pmatrix} = (\frac{11}{5}, \frac{2}{5}, \frac{2}{5}) \geq (\frac{2}{5}, \frac{2}{5}, \frac{2}{5})$

$$A\vec{q}^T = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & -2 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 3/5 \\ 2/5 \end{pmatrix} = \begin{pmatrix} 2/5 \\ 2/5 \\ 1/5 \end{pmatrix} \leq \begin{pmatrix} 2/5 \\ 2/5 \\ 2/5 \end{pmatrix}$$